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Heisenberg Equations of Motion for Spin-$\frac{1}{2}$ Wave Equation in General Relativity

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Abstract

The Heisenberg equations of motion for the spin-$\frac{1}{2}$ wave equation in general relativity are obtained by a covariant procedure. They are found to be similar to the equations of motion for a classical pole-dipole test-particle in general relativity. The identification is complete when the Heisenberg equations are taken to be satisfied by the respective expectation values.

1. *Introduction*

Although the spin-$\frac{1}{2}$ wave equation in general relativity has existed for a long time (Weyl, 1929, Fock, 1929), little yet is known of its physical contents. In this paper we shall obtain the equations of motion in the Heisenberg picture and thereby suggest a link with the classical spinning test-particle in general relativity.

While it is known that for the spin-$\frac{1}{2}$ wave equation a hermitean Hamiltonian exists (Oliveira & Tiomno, 1962) so that one can go over to the Heisenberg picture without any problem, the loss of manifest general covariance in such a procedure makes the interpretation of the subsequent equations rather difficult. Instead we shall use a device which treats the time-coordinate as a canonical variable on the same footing as the space-coordinates but with the permissible state-vectors restricted by a supplementary condition. In this way, covariant equations of motion for the coordinates, their conjugate momenta, and the angular momentum of the particle can be written down by straight-forward computations. When these equations are taken to be satisfied by the expectation values of the respective quantities, they are completely identical to the equations for a classical

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pole-dipole test-particle in general relativity first obtained by Papapetrou (1951), with the supplementary condition for the angular momentum that of Pirani's (1956) (see also Taub, 1964).

2. *Spin-½ Wave Equation in General Relativity*

In this section we shall establish the notations and collect together some facts which will be useful for later discussions.

We take the point of view that spinors in an arbitrary Riemannian space are defined with respect to a pseudo-orthonormal tetrad (the Vierbeine) $e_i{}^\mu$ satisfying

$$g_{\mu\nu} e_i{}^\mu e_j{}^\nu = \eta_{ij} \tag{2.1}$$

where η_{ij} is the constant matrix $(1,-1,-1,-1)$. The notation to be used will be the common one in which Latin indices denote components with respect to the tetrad and Greek indices the space-time components. Raising and lowering of the Latin indices by η_{ij} and its inverse η^{ij} will be allowed. Conversion from the Latin indices to the Greek or *vice versa* is effected by multiplication with an appropriate number of vectors from the tetrad.

Before writing the spin-½ wave equation in general relativity (or the Dirac equation in an arbitrary coordinate system, which differs from it only in the vanishing of the Riemann-Christoffel curvature tensor) let us recapitulate the tetrad formulation in an external gravitational field. In this formulation, all dynamical quantities (other than the gravitational field) are to be written in terms of their tetrad-components. Equations satisfied by them are to be covariant, both under general coordinate transformations and independent rotations of the tetrads at different points. The conversion from the usual formulation to the tetrad formulation is straight-forward except for the covariant derivatives. For example, to rewrite the covariant derivative $\nabla_\mu T^\nu$ of a vector field T^ν in terms of its tetrad-components T^i, we multiply $\nabla_\mu T^\nu$ by $e^i{}_\nu$ and write

$$\begin{aligned} e^i{}_\nu \nabla_\mu T^\nu &= \nabla_\mu(e^i{}_\nu T^\nu) - (\nabla_\mu e^i{}_\nu) T^\nu \\ &= \partial_\mu T^i - \Gamma^i{}_{j\mu} T^j \\ &= i\text{th component of } [\partial\mu + (i/2)\Gamma^{kl}{}_\mu M_{kl}]T \end{aligned}$$

where

$$\Gamma^i{}_{j\mu} = e_j{}^\nu \nabla_\mu e^i{}_\nu \tag{2.2}$$

and in the last line, T is regarded as a column vector and M_{kl} 4×4 matrices satisfying

$$[M_{ij}, M_{kl}] = iC_{ij,kl,}{}^{pq} M_{pq} \tag{2.3}$$

with

$$C_{ij,kl,}{}^{pq} = \eta_{il}\,\delta_j{}^{[p}\,\delta_k{}^{q]} + \eta_{jk}\,\delta_i{}^{[p}\,\delta_l{}^{q]} - \eta_{ik}\,\delta_j{}^{[p}\,\delta_l{}^{q]} - \eta_{jl}\,\delta_i{}^{[p}\,\delta_k{}^{q]} \tag{2.4}$$

In general, covariant differentiation ∇_μ of a tensor of any rank is to be replaced by the operation

$$d\mu = \partial_\mu + \frac{i}{2}\Gamma^{kl}{}_\mu M_{kl} \tag{2.5}$$

with M_{kl} forming a set of generators for the respective representation of the homogeneous Lorentz group to which the tetrad-components of the tensor belong. The commutation relation (2.3) is universal.

With these preliminaries, we can immediately write down the spin-½ wave equation in general relativity as a direct generalisation of the Dirac equation as follows:

$$i\hbar e_i{}^{\mu}\gamma^i\left(\partial_\mu + \frac{i}{2}\Gamma^{kl}{}_{\mu}M_{kl}\right)\psi - m\psi = 0 \tag{2.6}$$

where the γ's are 4×4 matrices satisfying

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij} \tag{2.7}$$

and

$$M_{ij} = \frac{i}{4}[\gamma_i, \gamma_j] \tag{2.8}$$

are generators of the homogeneous Lorentz group in the representation by four-component spinors.

The equation is obviously covariant under coordinate transformations if we remember that the individual components of ψ, like the tetrad-components of a tensor, transform as scalars. To prove covariance under point-dependent rotations of the tetrads, consider the infinitesimal transformation

$$\delta e_i{}^{\mu} = \omega_i{}^j e_j{}^{\mu} \tag{2.9}$$

where $\omega_i{}^j$ are arbitrary functions of space-time subject only to the condition

$$\omega_{ij} = -\omega_{ji} \tag{2.10}$$

Straight-forward computation from (2.2) shows that

$$\delta\Gamma^{ij}{}_{\mu} = \omega^i{}_k\Gamma^{kj}{}_{\mu} + \omega^j{}_k\Gamma^{ik}{}_{\mu} + \partial_\mu\omega^{ij} \tag{2.11}$$

Together with the postulated transformation law for the spinor

$$\delta\psi = -\frac{i}{2}\omega^{ij}M_{ij}\psi \tag{2.12}$$

we can then establish

$$\delta d_\mu\psi = -\frac{i}{2}\omega^{ij}M_{ij}d_\mu\psi \tag{2.13}$$

Finally, making use of the commutation relations

$$[\gamma^i, M_{jk}] = i(\delta^i{}_j\gamma_k - \delta^i{}_k\gamma_j) \tag{2.14}$$

we can show that the change in the first term of (2.6) is

$$\delta(i\hbar\gamma^\mu d_\mu\psi) = -\frac{i}{2}\omega^{ij}M_{ij}(i\hbar\gamma^\mu d_\mu\psi) \tag{2.15}$$

Since the last term of (2.6) transforms in like manner, the covariance of (2.6) is then established.

Note that we have here a structure which is identical to that of the usual gauge-invariant theories, with the homogeneous Lorentz group acting as the gauge group, and $\Gamma^{ij}{}_{\mu}$ the gauge field. The quantity $R^{ij}{}_{\mu\nu}$ obtained from the definition

$$[d_{\mu}, d_{\nu}] = -\frac{i}{2}R^{ij}{}_{\mu\nu}M_{ij} \tag{2.16}$$

as

$$R^{ij}{}_{\mu\nu} = \partial_{\nu}\Gamma^{ij}{}_{\mu} - \partial_{\mu}\Gamma^{ij}{}_{\nu} + \tfrac{1}{2}C_{ij,kl,}{}^{pq}\Gamma^{ij}{}_{\mu}\Gamma^{kl}{}_{\nu} \tag{2.17}$$

is the analogue of the electromagnetic field $f_{\mu\nu}$, and transforms homogeneously under the gauge group. Furthermore, it is just equal to the Riemann–Christoffel curvature tensor with two of its Greek indices converted into Latin indices, i.e.

$$R^{ij}{}_{\mu\nu} = e^{i}{}_{\lambda}e^{j}{}_{\rho}R^{\lambda\rho}{}_{\mu\nu} \tag{2.18}$$

This justifies our choice of notation in (2.16). We leave to Appendix A for a proof of it.

3. *Heisenberg Equations of Motion*

The equation (2.6) gives the time-development of the wave function ψ in the Schrödinger picture. It can be written in the standard form of the Schrödinger equation with a hermitean Hamiltonian (De Oliveira & Tiomno, 1962) and hence transformed to the Heisenberg picture in which the dynamical operators are given time-dependence. Such a procedure, however, will lead to non-covariant equations which are difficult to compare with the classical theory. We shall try to maintain explicit covariance by considering instead an enlarged system in which t and $(\hbar/i)/\partial/\partial t$ are introduced as a pair of conjugate dynamical operators, and whose 'dynamics' is understood as the dependence on a parameter τ of the state-vector $\psi(x^{\mu};\tau)$ through the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial\tau} = \mathscr{H}\psi \tag{3.1}$$

where

$$\mathscr{H} = \gamma^{\mu}\pi_{\mu} + m \tag{3.2}$$

with

$$\pi_{\mu} = \frac{\hbar}{i}d_{\mu} = p_{\mu} + \tfrac{1}{2}\Gamma^{ij}{}_{\mu}S_{ij} \tag{3.3}$$

$$p_{\mu} = \frac{\hbar}{i}\partial_{\mu} \tag{3.4}$$

$$S_{ij} = \hbar M_{ij} \tag{3.5}$$

$\mathscr{H}$ is in fact an operator whose null space is identical to the solution space of (2.6). As shown in Appendix B, from the Heisenberg equation of motion

$$i\hbar \frac{dO}{d\tau} = [O, \mathscr{H}] \tag{3.6}$$

for any operator O in this generalised Heisenberg picture, one can recover the Heisenberg equations for the original system by operating both sides of (3.6) on vectors ψ satisfying the constraint equation

$$\mathscr{H}\psi = 0 \tag{3.7}$$

provided we are allowed to replace expectation values of products of operators by the products of expectation values, an assumption we shall adopt in this paper.

We shall now write down the Heisenberg equations for the operators x^μ, π_μ, and S_{ij}. For x^μ, it is

$$\frac{dx^\mu}{d\tau} = \frac{1}{i\hbar}[x^\mu, \mathscr{H}] = \gamma^\mu \tag{3.8}$$

use having been made of the commutation relation

$$[x^\mu, p_\nu] = i\hbar\delta^\mu{}_\nu \tag{3.9}$$

For π_μ, we have

$$\begin{aligned}\frac{d\pi_\mu}{d\tau} &= \frac{1}{i\hbar}[\pi_\mu, \mathscr{H}] = \frac{1}{i\hbar}[\pi_\mu, \gamma^\nu]\pi_\nu + \frac{1}{i\hbar}\gamma^\nu[\pi_\mu, \pi_\nu] \\ &= \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix}\gamma^\lambda \pi_\nu + \tfrac{1}{2}R^{\lambda\rho}{}_{\mu\nu}\gamma^\nu S_{\lambda\rho}\end{aligned} \tag{3.10}$$

where we have used

$$[\pi_\mu, \pi_\nu] = \frac{i\hbar}{2}R^{ij}{}_{\mu\nu}S_{ij} = \frac{i\hbar}{2}R^{\lambda\rho}{}_{\mu\nu}S_{\lambda\rho} \tag{3.11}$$

which is a direct transcription of (2.16) using also (2.18), and

$$\begin{aligned}\frac{1}{i\hbar}[\pi_\mu, \gamma^\nu] &= -[d_\mu, e_i{}^\nu\gamma^i] \\ &= -(\partial_\mu e_i{}^\nu)\gamma^i - \frac{i}{2}\Gamma^{ij}{}_\nu[M_{ij}, \gamma^k]e_k{}^\nu \\ &= -(\partial_\mu e_i{}^\nu + \Gamma^j{}_{i\mu}e_j{}^\nu)\gamma^i \\ &= -[\partial_\mu e_i{}^\nu + (\nabla_\mu e^j{}_\rho)e_i{}^\rho e_j{}^\nu]\gamma^i \\ &= -(\partial_\mu e_i{}^\nu - \nabla_\mu e_i{}^\nu)\gamma^i = \begin{Bmatrix} \nu \\ \lambda\mu \end{Bmatrix}\gamma^\lambda\end{aligned} \tag{3.12}$$

Finally, for S_{ij} we obtain

$$\begin{aligned}\frac{dS_{ij}}{d\tau} &= \frac{1}{i\hbar}[S_{ij}, \mathscr{H}] = \frac{1}{i\hbar}[S_{ij}, \gamma^k]\pi_k + \frac{1}{i\hbar}\gamma^\mu[S_{ij}, \pi_\mu] \\ &= \gamma_i\pi_j - \gamma_j\pi_i + \tfrac{1}{2}C_{ij,kl,}{}^{pq}\Gamma^{kl}{}_\mu\gamma^\mu S_{pq}\end{aligned} \tag{3.13}$$

when use is made of

$$[S_{ij}, \gamma^k] = -i\hbar(\delta^k{}_i \gamma_j - \delta^k{}_j \gamma_i) \tag{3.14}$$

and

$$[S_{ij}, \pi_\mu] = \tfrac{1}{2} i\hbar C_{ij,kl,}{}^{pq} \Gamma^{kl}{}_\mu S_{pq} \tag{3.15}$$

which are immediate consequences of (2.14) and (2.3) respectively.

The equations (3.8), (3.10) and (3.13) are to be supplemented by the constraint equation

$$\gamma^\mu \pi_\mu + m \approx 0 \tag{3.16}$$

which is a weak equation in the sense of Dirac's (1964). As a simple consequence of the algebraic properties of the γ-matrices, we also have

$$\{S_{\mu\nu}, \gamma^\nu\} = 0 \tag{3.17}$$

If we write $u^\mu \equiv dx^\mu/d\tau$ for γ^μ and regard the subsequent equations as satisfied by the expectation values, we then obtain from (3.16)

$$u^\mu \pi_\mu + m = 0 \tag{3.16a}$$

from (3.10),

$$\frac{D\pi_\mu}{D\tau} - \tfrac{1}{2} R^{\lambda\rho}{}_{\mu\nu} u^\nu S_{\lambda\rho} = 0 \tag{3.10a}$$

where

$$\frac{D\pi_\mu}{D\tau} \equiv \frac{d\pi_\mu}{d\tau} - \begin{Bmatrix} \lambda \\ \nu\mu \end{Bmatrix} u^\nu \pi_\lambda \tag{3.18}$$

from (3.17),

$$S_{\mu\nu} u^\nu = 0 \tag{3.17a}$$

and from (3.13) after multiplication by $e^i{}_\mu e^j{}_\nu$

$$e^i{}_\mu e^j{}_\nu \left(\frac{dS_{ij}}{d\tau} - \tfrac{1}{2} C_{ij,kl,}{}^{pq} \Gamma^{kl}{}_\mu u^\mu S_{pq} \right) = u_\mu \pi_\nu - u_\nu \pi_\mu$$

The left-hand side of the last equation can be written as $DS_{\mu\nu}/D\tau$, for

$$\begin{aligned}
\frac{DS_{\mu\nu}}{D\tau} &\equiv \frac{dS_{\mu\nu}}{d\tau} - \begin{Bmatrix} \lambda \\ \mu\rho \end{Bmatrix} u^\rho S_{\lambda\nu} - \begin{Bmatrix} \lambda \\ \nu\rho \end{Bmatrix} u^\rho S_{\mu\lambda} \\
&= e^i{}_\mu e^j{}_\nu \frac{dS_{ij}}{d\tau} + \left[\frac{d}{d\tau}(e^i{}_\mu e^j{}_\nu) - \begin{Bmatrix} \lambda \\ \mu\rho \end{Bmatrix} u^\rho e^i{}_\lambda e^j{}_\nu - \begin{Bmatrix} \lambda \\ \nu\rho \end{Bmatrix} u^\rho e^i{}_\mu e^j{}_\lambda \right] S_{ij} \\
&= e^i{}_\mu e^j{}_\nu \frac{dS_{ij}}{d\tau} + (e^j{}_\nu \nabla\rho\, e^i{}_\mu + e^i{}_\mu \nabla_\rho e^j{}_\nu) u^\rho S_{ij} \\
&= e^i{}_\mu e^j{}_\nu \left(\frac{dS_{ij}}{d\tau} + \Gamma^k{}_{i\rho} u^\rho S_{kj} + \Gamma^k{}_{j\rho} u^\rho S_{ik} \right)
\end{aligned}$$

So we have for the angular momentum

$$\frac{DS_{\mu\nu}}{D\tau} - u_\mu \pi_\nu + u_\nu \pi_\mu = 0 \tag{3.13a}$$

The equations (3.10a), (3.13a), (3.16a) and (3.17a) are very similar to the equations of motion of a classical pole-dipole test-particle in an external gravitational field. To complete the identification, we shall show that $u^2 \equiv u_\mu u^\mu$ is a constant of motion from these equations, which fact will then enable us to define the proper time of the particle through a normalisation of the velocity. To do this, we multiply (3.13a) by u^ν and make use of (3.16a) to obtain

$$u^2 \pi_\mu = -mu_\mu - \frac{DS_{\mu\nu}}{D\tau} u^\nu$$

Operating on this equation with $u^\mu(D/D\tau)$, using again (3.16a) and $u^\mu(D\pi_\mu/D\tau) = 0$, which is the consequence of (3.10a), we then have

$$-m\frac{Du^2}{D\tau} = -\tfrac{1}{2}m\frac{Du^2}{D\tau} + u^\mu \frac{D}{D\tau}\left(\frac{DS_{\mu\nu}}{D\tau} u^\nu\right)$$

The last term being zero by using (3.17a) and the antisymmetry of $S_{\mu\nu}$, we are left with

$$\frac{Du^2}{D\tau} = 0$$

and consequently

$$u^2 = \frac{1}{\kappa^2}$$

where κ is a constant of motion. Defining $S = (1/\kappa)\tau$, writing $U^\mu = dx^\mu/ds$, $m' = \kappa m$ and eliminating π_μ from equations (3.10a), (3.13a), (3.16a) and (3.17a) through the equation

$$\pi_\mu = -m' U_\mu - \frac{DS_{\mu\nu}}{Ds} U^\nu \tag{3.19}$$

we obtain finally†

$$\frac{D}{Ds}\left(m' U_\mu + \frac{DS_{\mu\nu}}{DS} U^\nu\right) + \tfrac{1}{2}R^{\lambda\rho}{}_{\mu\nu} u^\nu S_{\lambda\rho} = 0 \tag{3.20}$$

$$\frac{DS_{\mu\nu}}{Ds} + U_\mu U^\rho \frac{DS_{\nu\rho}}{Ds} - U_\nu U^\rho \frac{DS_{\mu\rho}}{Ds} = 0 \tag{3.21}$$

$$S_{\mu\nu} U^\nu = 0 \tag{3.22}$$

$$U_\mu U^\mu = 1 \tag{3.23}$$

This set is completely identical to the equations obtained by Papapetrou (1951) and Pirani (1956) by taking the limit of an extended classical source of energy-momentum.

† Had we started with (2.6) but with m replaced by $-m$ corresponding merely to a different convention for the usual Dirac equation, we would have arrived at the same equations but with $S_{\mu\nu}$ replaced by $-S_{\mu\nu}$. It is not clear that these two systems of equations, though mathematically different, can be distinguished physically.

4. *Remarks*

We should of course hasten to add that the above argument is essentially formal in character. In the absence of detailed investigation of the conditions of validity of the approximation, it is hard to conclude whether the Heisenberg equations of motion are really satisfied by the expectation values. In particular, we have avoided including the equation

$$\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g^{\mu\nu}$$

suggested by the commutation relation $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$, which would lead to unwanted restriction on the velocity. In this sense, the argument should be taken as suggesting a connection of the spin-$\frac{1}{2}$ wave equation to the equations for a classical pole-dipole particle rather than as an independent derivation of the latter.

It is perhaps rather satisfying to note that in the particular case of special relativity, when the Riemann–Christoffel curvature tensor vanishes, the equations (3.20–3.23) can accommodate, through the arbitrariness of m', the motions of all the different local centres of mass of the particle (Hönl and Papapetrou, 1939; Møller, 1952; see also Schild, 1967). We may regard the usual Dirac equation as carrying with it this content.

Finally, it would be interesting to generalise the present considerations to wave equations of arbitrarily high spin in general relativity, for this may open up, like the case with isotropic spin (Wong, 1970), the possibility to consider the limit when $\hbar$ tends to zero but the spin tends to infinity in such a way that a finite angular momentum survives in the classical limit.

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Appendix A

To prove $R^{ij}{}_{\mu\nu} = e^i{}_\lambda e^j{}_\rho R^{\lambda\rho}{}_{\mu\nu}$, we start from the Ricci identity for an arbitrary vector field T^μ, viz.,

$$2\nabla_{[\mu}\nabla_{\nu]}T^\lambda = R^\lambda{}_{\rho\mu\nu}T^\rho \tag{A.1}$$

where

$$R^\lambda{}_{\rho\mu\nu} = \partial_\mu\begin{Bmatrix}\lambda\\ \rho\nu\end{Bmatrix} - \partial_\nu\begin{Bmatrix}\lambda\\ \rho\mu\end{Bmatrix} + \begin{Bmatrix}\lambda\\ \tau\mu\end{Bmatrix}\begin{Bmatrix}\tau\\ \rho\nu\end{Bmatrix} - \begin{Bmatrix}\lambda\\ \tau\nu\end{Bmatrix}\begin{Bmatrix}\tau\\ \rho\mu\end{Bmatrix} \tag{A.2}$$

and we shall multiply both sides of (A.1) by $e^i{}_\lambda$. Noting from Section 2 that

$$\nabla_\nu T^\lambda = e_i{}^\lambda d_\nu T^i \tag{A.3}$$

we have

$$\begin{aligned} e^i{}_\lambda \nabla_\mu \nabla_\nu T^\lambda &= \nabla_\mu(e^i{}_\lambda \nabla_\nu T^\lambda) - \nabla_\mu e^i{}_\lambda \nabla_\nu T^\lambda \\ &= \partial_\mu(d_\nu T^i) - \begin{Bmatrix} \rho \\ \nu\mu \end{Bmatrix} d_\rho T^i - (\nabla_\mu e^i{}_\lambda) e_j{}^\lambda d_\nu T^j \\ &= \partial_\mu(d_\nu T^i) - \Gamma^i{}_{j\mu} d_\nu T^j - \begin{Bmatrix} \rho \\ \nu\mu \end{Bmatrix} d_\rho T^i \\ &= d_\mu d_\nu T^i - \begin{Bmatrix} \rho \\ \nu\mu \end{Bmatrix} d_\rho T^i \end{aligned}$$

so that the left-hand side of (A.1), on multiplication by $e^i{}_\lambda$, becomes $[d_\mu, d_\nu]T^i$, which equals $R^{ij}{}_{\mu\nu}T_j$ if we make use of (2.16), choosing M_{ij} as generators for the vector-representation of the homogeneous Lorentz group. On the other hand, the right-hand side of (A.1) becomes $e^i{}_\lambda R^{\lambda\rho}{}_{\mu\nu} e^j{}_\rho T_j$. Hence $R^{ij}{}_{\mu\nu} = e^i{}_\lambda e^j{}_\rho R^{\lambda\rho}{}_{\mu\nu}$.

Appendix B

To justify the device employed in Section 3 for obtaining the Heisenberg equations of motion through an enlargement of the space of dynamical operators, consider a quantum mechanical system with a set of canonical coordinates collectively denoted by q and conjugate momenta p, with a Hamiltonian operator $H(q,p,t)$. The wave function $\psi(q,t)$ satisfies

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{B.1}$$

Consider an enlarged system in which in addition to q and p, t and π form also a pair of conjugate variables satisfying

$$[t, \pi] = i\hbar \tag{B.2}$$

and the wave function $\Psi(q, t; \tau)$ satisfies

$$i\hbar \frac{\partial \Psi}{\partial t} = \mathscr{H}\Psi \tag{B.3}$$

where

$$\mathscr{H} \equiv f(H + \pi) \tag{B.4}$$

f being an arbitrary operator.

If we require Ψ to satisfy the supplementary condition

$$(H + \pi)\Psi = 0 \tag{B.5}$$

then the resulting system is equivalent to the original one in the Schrödinger picture. In the (generalised) Heisenberg picture, if O is an operator depending only on q and p,

$$i\hbar \frac{dO}{d\tau} = [O, \mathscr{H}] = [O, f](H + \pi) + f[O, H] \approx f[O, H] \tag{B.6}$$

where $\approx$ means equality when restricted to state-vectors satisfying (B.5). Also,

$$\frac{dt}{d\tau} = \frac{1}{i\hbar}[t,f](H+\pi) + \frac{1}{i\hbar}f[t,\pi] \approx f \tag{B.7}$$

The method of Section 3 amounts to a proposal to use (B.6) and (B.7) in place of

$$i\hbar\frac{dO}{dt} = [O,H] \tag{B.8}$$

for the original system. Though as operator equations they are not quite identical, under the assumption we made of replacing expectation values of operator products by products of expectation values, we have from (B.6) and (B.7)

$$\begin{aligned} i\hbar\left\langle\frac{dO}{d\tau}\right\rangle &= \langle f\rangle\langle[O,H]\rangle \\ &= \left\langle\frac{dt}{d\tau}\right\rangle\langle[O,H]\rangle \end{aligned}$$

whence

$$i\hbar\left\langle\frac{dO}{dt}\right\rangle = \langle[O,H]\rangle$$

This shows that the generalised Heisenberg equations of motion plus the assumption on expectation values are equivalent to the correct Heisenberg equations plus the same assumption.

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